

Restricted q -Isometry Properties Adapted to Frames for Nonconvex l_q -Analysis *

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Abstract

This paper discusses reconstruction of signals from few measurements in the situation that signals are sparse or approximately sparse in terms of a general frame via the l_q -analysis optimization with $0 < q \leq 1$. We first introduce a notion of restricted q -isometry property (q -RIP) adapted to a dictionary, which is a natural extension of the standard q -RIP, and establish a generalized q -RIP condition for approximate reconstruction of signals via the l_q -analysis optimization. We then determine how many random, Gaussian measurements are needed for the condition to hold with high probability. The resulting sufficient condition is met by fewer measurements for smaller q than when $q = 1$.

The introduced generalized q -RIP is also useful in compressed data separation. In compressed data separation, one considers the problem of reconstruction of signals'

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distinct subcomponents, which are (approximately) sparse in morphologically different dictionaries, from few measurements. With the notion of generalized q -RIP, we show that under an usual assumption that the dictionaries satisfy a mutual coherence condition, the l_q split analysis with $0 < q \leq 1$ can approximately reconstruct the distinct components from fewer random Gaussian measurements with small q than when $q = 1$.

Keywords. Compressed sensing, Restricted isometry property, Frames, l_q -analysis, Sparse recovery, Data separation.

1 Introduction

1.1 Background

Recovery of signals which are (approximately) sparse in terms of a dictionary from few measurements is one of the major subjects in compressed sensing. Suppose that we observe data from the model

$$y = Af,$$

where $A \in \mathbb{R}^{m \times n}$ (with $m < n$) is a known measurement matrix. Our goal is to reconstruct the unknown signal f based on y and A .

In standard compressed sensing [10, 9, 16], one assumes that f is sparse in the standard coordinate basis. A vector v is s sparse if it has at most s nonzero entries. If the measurement matrix A satisfies a restricted isometry property (RIP) condition $\delta_{cs} \leq C$ (see e.g. [9, 6] and the references therein), one can recover a sparse signal f by solving an l_1 -minimization problem

$$\min_{\tilde{f} \in \mathbb{R}^n} \|\tilde{f}\|_1 \quad \text{subject to} \quad A\tilde{f} = y. \quad (L_1)$$

Recall that a matrix A is said to satisfy the RIP [10] of order s if there is some $\delta \in [0, 1)$

such that, for all x with $\|x\|_0 \leq s$, we have

$$(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2.$$

The infimum of all possible δ satisfying the above inequality, denoted as δ_s , is the so-called RIP constant of order s . Many types of random measurement matrices such as Gaussian matrices or Sub-Gaussian matrices have the RIP constant $\delta_s \leq \delta$ with overwhelming probability provided that $m \geq C\delta^{-2}s \log(n/s)$ [10, 3, 32, 34]. Based on its RIP guarantees, with high probability, (L_1) can recover every s sparse vector from $O(s \log(n/s))$ random measurements.

One alternative way of finding the unknown signal proposed in the literature is to solve

$$\min_{\tilde{f} \in \mathbb{R}^n} \|\tilde{f}\|_q \quad \text{subject to} \quad A\tilde{f} = y. \quad (L_q)$$

Reconstructing sparse signals via (L_q) with $0 < q < 1$ has been considered in a series of papers (see e.g. [12, 35, 22, 15, 24] and the references therein) and some of the virtues are highlighted recently. The l_q -strategy offers an advantage in that it requires fewer measurements in numerical experiments [11], with random and nonrandom Fourier measurements. Chartrand and Staneva [12] showed that if A is an $m \times n$ Gaussian matrix, every s sparse vector f can be exactly recovered by solving (L_q) with high probability provided

$$m \geq C_1(q)s + qC_2(q)s \log(n/s),$$

where $C_1(q)$ and $C_2(q)$ are bounded and given explicitly there. The dependence of m on the number n of columns vanishes for $q \rightarrow 0$. In their proof, they used a restricted q -isometry property, namely

$$(1 - \delta)\|v\|_2^q \leq \|Av\|_q^q \leq (1 + \delta)\|v\|_2^q$$

for all s sparse vectors $v \in \mathbb{R}^n$ and $0 < q \leq 1$.

In this paper, the signal is assumed to be (approximately) sparse in terms of a frame D , i.e., D^*f is (approximately) sparse. Some examples in practice are Gabor frames [20] in

radar and sonar, curvelet frames [7] and undecimated wavelet transforms [31, 5] in image processing, etc. Recall that the columns of $D \in \mathbb{R}^{n \times d}$ ($n \leq d$) form a frame for \mathbb{R}^n with frame bounds $0 < \mathcal{L} \leq \mathcal{U} < \infty$ if

$$\forall f \in \mathbb{R}^n, \quad \mathcal{L}\|f\|_2^2 \leq \|D^*f\|_2^2 \leq \mathcal{U}\|f\|_2^2. \quad (1.1)$$

If $\mathcal{U} = \mathcal{L}$, then D is a tight frame for \mathbb{R}^n . One way of recovery such signals is via the following l_q analysis (see e.g. [18] and the reference therein) with $0 < q \leq 1$:

$$\hat{f} = \arg \min_{\tilde{f} \in \mathbb{R}^n} \|D^*\tilde{f}\|_q \quad \text{subject to} \quad A\tilde{f} = y. \quad (P_q)$$

We remark that (P_q) may have more than one minimizer, and our results of this paper hold for any solution of (P_q) . Here, for simplicity of statements, we assume that (P_q) has a unique minimizer. Letting D be a tight frame, Candès et al. [8] showed that the solution \hat{f} of (P_1) satisfies

$$\|\hat{f} - f\|_2 \leq C_0 \frac{\|D^*f - (D^*f)_{[s]}\|_1}{\sqrt{s}},$$

provided that A satisfies an D -RIP condition. Here we denote $x_{[s]}$ to be the vector consisting of the s largest coefficients of $x \in \mathbb{R}^d$ in magnitude:

$$x_{[s]} = \arg \min_{\|\tilde{x}\|_0 \leq s} \|x - \tilde{x}\|_2.$$

Recall that a measurement matrix A is said to obey the restricted isometry property adapted to D (abbreviated as D -RIP) [8] of order s if there exists some $\delta \in (0, 1)$ such that

$$(1 - \delta)\|Dv\|_2^2 \leq \|ADv\|_2^2 \leq (1 + \delta)\|Dv\|_2^2 \quad (1.2)$$

holds for all s sparse vectors $v \in \mathbb{R}^d$. The D -RIP constant of order s , denoted as δ_s , is the infimum of all possible δ satisfying the above inequality. Note that the D -RIP is a natural extension of the standard RIP. Under the assumption that A satisfies an D -RIP condition, for general $0 < q \leq 1$, [1, 27] provided results on recovery of signals which are compressible in

terms of a tight frame D via (P_q) . Liu et al. [29] considered the problem of recovering signals which are compressible in a general frame D via dual frame based l_1 -analysis model. Nam et al. [33] proposed a new signal model called cospase analysis model with corresponding reconstruction methods. In a recent paper, Rauhut and Kabanava [25] provided both uniform and nonuniform recovery guarantees from Gaussian random measurements, which requires $O(s \log(d/s))$ measurements, for cospase signals based on (P_1) when D is a frame.

The D -RIP is a special case of a more general definition given in [2, 30]. Until now, nearly all good constructions of D -RIP measurement matrices uses randomness. For any choice of $D \in \mathbb{R}^{n \times d}$, if A is populated with independent and identically distributed (i.i.d.) random entries from a Gaussian or Sub-Gaussian distribution, then with high probability, A will satisfy the D -RIP of order s as long as $m = O(s \log(d/s))$ [8, 2, 30]. In fact, given any matrix A satisfying the traditional RIP, by applying a random sign matrix one obtains a matrix satisfying the D -RIP [26]. Based on its D -RIP guarantees, the aforementioned results show that (P_q) with $0 < q \leq 1$ can guarantee approximately recovery from $O(s \log(d/s))$ measurements for Sub-Gaussian matrices.

1.2 Main contribution

In this paper, we further develop theoretical results on l_q analysis for approximate recovery of signals, that are approximately sparse with respect to a general frame D . One of our main results shows that (P_q) can approximately recover the unknown signal with high probability from fewer measurements with small q than that were needed in the aforementioned results. Concretely, we have the following result.

Theorem 1.1. *Suppose that we observe data from the model $y = Af$. Let $D \in \mathbb{R}^{n \times d}$ be a frame with frame bounds $0 < \mathcal{L} \leq \mathcal{U} < \infty$. Let A be an $m \times n$ matrix whose entries are i.i.d. random distributed normally with mean zero and variance σ^2 . Then there exist constants*

$C_1(q)$ and $C_2(q)$ such that whenever $0 < q \leq 1$ and

$$m \geq C_1(q) \kappa^{\frac{q}{2-q}} s + q C_2(q) \kappa^{\frac{2q}{2-q}} s \log(d/s), \quad \kappa = \mathcal{U}/\mathcal{L},$$

with probability exceeding $1 - 1/\binom{d}{s}$, any solution \hat{f} of (P_q) satisfies

$$\|\hat{f} - f\|_2 \leq C \frac{\|D^* f - (D^* f)_{[s]}\|_q}{s^{1/q-1/2}}.$$

Remark 1.2. 1) $C_1(q)$ and $C_2(q)$ are bounded positive numbers which will be explicitly given in the proof of the theorem.

2) The dependence of m on the number d and the condition number κ of D vanishes for $q \rightarrow 0$. As a result, the required measurements become Cs when q is small, which are fewer than that were needed in the previous results.

3) Using the proof techniques developed in [36], one can improve the success probability.

The proof of this result is based on a notion of (D, q) -RIP and general (D, q) -RIP recovery result. It is a natural extension of the standard q -RIP in [12] (and for $q = 1$, in [16]):

Definition 1.3 $((D, q)$ -RIP). Let D be an $n \times d$ matrix. A measurement matrix A is said to obey the restricted q -isometry property adapted to D (abbreviated as (D, q) -RIP) of order s with constant $\delta \in [0, 1)$ if

$$(1 - \delta) \|Dv\|_2^q \leq \|ADv\|_q^q \leq (1 + \delta) \|Dv\|_2^q \quad (1.3)$$

holds for all s sparse vectors $v \in \mathbb{R}^d$. The (D, q) -RIP constant δ_s is defined as the smallest number δ such that (1.3) holds for all s sparse vectors $v \in \mathbb{R}^d$.

In section 2, we first establish approximate recovery results for l_q analysis with the assumption that the measurement matrix A satisfies an (D^\dagger, q) -RIP condition. Here $D^\dagger =$

$(DD^*)^{-1}D$ is the canonical dual frame of D . Subsequently, we prove Theorem 1.1 by showing how many random Gaussian measurements are sufficient for the condition to hold with high probability. The resulting sufficient condition is met by fewer measurements for smaller q than when $q = 1$.

Our approach (P_q) with $q = 1$ is slightly different with the l_1 -analysis approach considered in [29, 21], i.e.,

$$\arg \min_{\tilde{f}} \|(D^\dagger)^* \tilde{f}\|_1 \quad \text{s.t.} \quad A\tilde{f} = y. \quad (1.4)$$

Using D (instead of D^\dagger) as analysis operator is preferable in some certain circumstances, e.g., when D is known while D^\dagger is hard to be known or computed in high dimension sparse recovery, or when D is of special structure which has fast computation algorithm while D^\dagger is not (see e.g. [14]). We also note that it is hard to verify the (D, q) -RIP (or D -RIP) for a deterministic measurement matrix, and for certain random measurement matrices, verifying the (D^\dagger, q) -RIP (or D^\dagger -RIP) is almost the same as verifying the (D, q) -RIP (or D -RIP).

The proof techniques in this paper may shed some lights on improving the existing D -RIP recovery results. Our proof for the (D^\dagger, q) -RIP guaranteeing results in Section 2.4 shows that a suitable (D^\dagger, q) -RIP condition implies the l_q null space property of order s relative to D (D -NSP $_q$) of the measurement matrix. Recall that the matrix A is said to satisfy the D -NSP $_q$ of order s [23, 13, 37, 1, 21] if there exists a constant θ with $\theta \in (0, 1)$ such that for all $h \in \ker A$ and for all sets $T \subset \{1, \dots, d\}$ with cardinality $|T| \leq s$,

$$\|D_T^* h\|_q^q \leq \theta \|D_{T^c}^* h\|_q^q.$$

Here, D_T is the submatrix of D formed from the columns of D indexed by T . The smallest value of the constant θ in the above is referred to as the D -NSP $_q$ constant. The importance of the null space property is that it is the necessary and sufficient condition under which l_q recovery is exact for s -sparse signals for the case $D = I$ (see e.g. [24, 13]). By developing a

tighter relationship between the D -RIP constant and the D -NSP constant, one can improve the RIP condition for the exact sparse recovery (see e.g. [6] for the case $D = I$ and [27] for the case of tight frames). For general frames case, it would be interesting to pursue a tighter relationship between the D^\dagger -RIP constant and the D -NSP constant, and then establish an D^\dagger -RIP recovery result for (P_1) using the approach of this paper. However, this is beyond the scope of this paper.

1.3 Compressed data separation

Numerous examples show that signals of interest might be classified as multimodal data, i.e., being composed of distinct subcomponents. One common task is to separate such data into appropriate single components for further analysis (e.g. [19, 4, 17, 5]). In [28], the authors considered data separation from few measurements, and showed that the two distinct subcomponents, assumed to be approximately s_1 and s_2 sparse in two dictionaries $D_1 \in \mathbb{R}^{n \times d_1}$ and $D_2 \in \mathbb{R}^{n \times d_2}$ respectively, can be approximately reconstructed by solving the l_1 split analysis, provided that the measurement matrix satisfies an D -RIP condition and the two dictionaries satisfy a mutual coherence (between the different dictionaries) condition. Based on the D -RIP analysis, under a mutual coherence condition between the two dictionaries, the l_1 split analysis can approximately reconstruct the distinct components from $O((s_1 + s_2) \log \frac{d_1 + d_2}{s_1 + s_2})$ random sub-Gaussian measurements. Refer to [28] and the references therein for more details on compressed data separation.

Our second contribution of this paper is to establish further theoretical recovery results for compressed data separation via l_q split analysis from random Gaussian measurements. With the (D, q) -RIP introduced in this paper, and under an usual assumption that the two dictionaries satisfy a mutual coherence condition, we show that the l_q split analysis with $0 < q \leq 1$ can approximately reconstruct the distinct components from fewer random

Gaussian measurements with small q than that are needed in previous results. Recall that the mutual coherence between two dictionaries [28] is defined as follow.

Definition 1.4. Let $D_1 = (d_{1i})_{1 \leq i \leq d_1}$ and $D_2 = (d_{2j})_{1 \leq j \leq d_2}$. The mutual coherence between D_1 and D_2 is defined as

$$\mu_1 = \mu_1(D_1; D_2) = \max_{i,j} |\langle d_{1i}, d_{2j} \rangle|.$$

We have the following result, whose proof will be given in Section 3 by applying a general theorem for compressed data separation where unknown signals are composed of ι ($\iota \geq 2$) components that are sparse in terms of r tight frames D_1, D_2, \dots, D_r .

Theorem 1.5. Suppose that we observe data from the model $y = A(f_1 + f_2)$. Let $D_1 \in \mathbb{R}^{n \times d_1}$ and $D_2 \in \mathbb{R}^{n \times d_2}$ be two arbitrary tight frames for \mathbb{R}^n with frame bound 1, respectively. Let A be an $m \times n$ matrix whose entries are i.i.d. random distributed normally with mean zero and variance σ^2 . Fix positive integers s_1 and s_2 . Assume that the mutual coherence μ_1 between D_1 and D_2 satisfies

$$\mu_1(s_1 + s_2) \left(\lceil (2^{3q/2} 5)^{\frac{2}{2-q}} \rceil + 1 \right) \left(\frac{1}{8 \cdot 5^{2/q}} + 1 \right) < 1.$$

Then there exist constants $C_1(q)$ and $C_2(q)$ such that whenever $0 < q \leq 1$ and

$$m \geq C_1(q)(s_1 + s_2) + qC_2(q)(s_1 + s_2) \log \left(\frac{d_1 + d_2}{s_1 + s_2} \right),$$

with probability exceeding $1 - 1/\binom{d_1 + d_2}{s_1 + s_2}$, any solution (\hat{f}_1, \hat{f}_2) to the l_q Split analysis

$$(\hat{f}_1, \hat{f}_2) = \arg \min_{\tilde{f}_1, \tilde{f}_2 \in \mathbb{R}^n} \|D_1^* \tilde{f}_1\|_q^q + \|D_2^* \tilde{f}_2\|_q^q \quad \text{s.t.} \quad A(\tilde{f}_1 + \tilde{f}_2) = y, \quad (1.5)$$

obeys

$$\|\hat{f}_1 - f_1\|_2 + \|\hat{f}_2 - f_2\|_2 \leq C_1 \frac{(\|D_1^* f_1 - (D_1^* f_1)_{[s_1]}\|_q^q + \|D_2^* f_2 - (D_2^* f_2)_{[s_2]}\|_q^q)^{1/q}}{(s_1 + s_2)^{1/q - 1/2}}. \quad (1.6)$$

Remark 1.6. *As q becomes smaller, weaker mutual coherence condition and fewer measurements are needed to guarantee approximate recovery. In particular, letting $q \rightarrow 0$, the mutual coherence condition and the required measurements become $2\mu_1(s_1 + s_2) < 1$ and $m = O(s_1 + s_2)$, respectively.*

As we will see in Section 3, Theorem 1.1 can be generalized to the cases where signals f are composed of general $\iota \in \mathbb{Z}^+$ distinct components. To the best of our knowledge, our results may be the first of this kind for a general ι . For simplicity, we have restricted to the tight frames case. Note that similar as Theorem 1.1, our result can be extend to the non-tight frames case.

The proof is similar to that of Theorem 1.1. Under the assumptions that the measurement matrix satisfies a generalized q -RIP condition, and that the dictionaries satisfy a mutual coherence condition, we first prove that the l_q split analysis with $0 < q \leq 1$ can approximately reconstruct the distinct components. Subsequently we determine how many random, Gaussian measurements are sufficient for the generalized q -RIP condition to hold with high probability. The resulting sufficient condition is met by fewer measurements for smaller q than when $q = 1$. Such a proof is given in Section 3.

1.4 Notation

For a vector $v \in \mathbb{R}^d$, $\|v\|_0$ is the number of nonzero entries of v . For any $q \in (0, \infty)$, denote $\|u\|_q = \left(\sum_{j=1}^d |u_j|^q\right)^{1/q}$ and $\|u\|_\infty = \max_j |u_j|$. For $d \in \mathbb{N}$, denote $[d]$ to mean $\{1, 2, \dots, d\}$. Given an index set $T \subset [d]$ and a matrix $D \in \mathbb{R}^{n \times d}$, T^c is the complement of T in $[d]$, D_T is the submatrix of D formed from the columns of D indexed by T .¹ For a matrix D_1 , we

¹We note that this notation will occasionally be abused to refer to the $n \times d$ matrix obtained by setting the columns of D indexed by T^c to zero. The usage should be clear from the context, but in most cases there is no substantive difference between the two.

write D_{1T} to mean $(D_1)_T$. Write D^* to mean the conjugate transpose of a matrix D , D^T to mean the transpose of D , and D_T^* to mean $(D_T)^*$. For a vector $x \in \mathbb{R}^d$, $x_{[s]}$ denotes the vector consisting of the s largest entries of x in magnitude. $C > 0$ (or c , c_1) denotes a universal constant that might be different in each occurrence. D^+ denotes the Moore-pseudo inverse of a matrix D , and $\ker D$ denotes the null space of D . For a frame D with frame bounds $0 < \mathcal{L} \leq \mathcal{U} < \infty$, $D^\dagger = (DD^*)^{-1}D$ is the canonical dual frame. Note that $D^\dagger = (D^*)^+$, $D^\dagger D^* = I$, and the lower and upper frame bound of D^\dagger is given by $1/\mathcal{U}$ and $1/\mathcal{L}$, respectively. The smallest and largest eigenvalues of a symmetric matrix $B \in \mathbb{R}^{d \times d}$ are denoted by $\lambda_{\min}(B)$ and $\lambda_{\max}(B)$, respectively.

2 Sparse recovery via l_q -analysis

In this section we prove Theorem 1.1. We first show that if the measurement matrix A satisfies an (D^\dagger, q) -RIP condition, then the unknown signal can be approximately recovered by solving the l_q analysis optimization. The following basic inequalities related to the l_p (quasi)norm are useful for our proofs. For any vectors $u, v \in \mathbb{R}^N$, one has

$$\|u\|_{p_2} \leq \|u\|_{p_1} \leq N^{1/p_1 - 1/p_2} \|u\|_{p_2}, \quad 0 < p_1 \leq p_2 \leq \infty \quad (2.1)$$

and the following triangle inequality for $\|\cdot\|_q^q$ with $q \in (0, 1]$:

$$\|u + v\|_q^q \leq \|u\|_q^q + \|v\|_q^q. \quad (2.2)$$

2.1 Recovery results based on (D^\dagger, q) -RIP

In this subsection, we give (D^\dagger, q) -RIP guarantee results on sparse recovery with frames from noisy measurements $y = Af + z$ via solving the following l_q -analysis optimization

$$\arg \min_{\tilde{f} \in \mathbb{R}^n} \|D^* \tilde{f}\|_q \quad \text{subject to} \quad \|A\tilde{f} - y\|_r \leq \varepsilon, \quad (2.3)$$

where $0 < q \leq 1, 1 \leq r \leq \infty, \varepsilon \geq 0$ and the noise term $z \in \mathbb{R}^m$ satisfies $\|z\|_r \leq \varepsilon$.

Theorem 2.1. *Let $0 < q \leq 1, 1 \leq r \leq \infty, \varepsilon \geq 0$. Suppose we observe data from the model $y = Af + z$ with $\|z\|_r \leq \varepsilon$. Let D be a frame with frame bounds $0 < \mathcal{L} \leq \mathcal{U} < \infty$. Fix positive integers s, a with $s < a$. Assume that the (D^\dagger, q) -RIP constant of the measurement matrix A satisfies*

$$\rho^{1-q/2} (\rho^{2/q-1} + 1)^{q/2} \kappa^q (1 + \delta_a) < 1 - \delta_{s+a}, \quad (2.4)$$

where

$$\rho = \frac{s}{a}, \quad \kappa = \frac{\mathcal{U}}{\mathcal{L}}. \quad (2.5)$$

Then any solution \hat{f} to (2.3) satisfies

$$\|\hat{f} - f\|_2 \leq C_1 \frac{\|D^* f - (D^* f)_{[s]}\|_q}{s^{1/q-1/2}} + C_2 m^{1/q-1/r} \varepsilon \quad (2.6)$$

and

$$\|D^* \hat{f} - D^* f\|_q^q \leq C_3 \|D^* f - (D^* f)_{[s]}\|_q^q + C_4 a^{1-q/2} m^{1-q/r} \varepsilon^q, \quad (2.7)$$

where C_1, C_2, C_3 and C_4 are positive constants (given explicitly in the proof) depending only on the (D^\dagger, q) -RIP constant $\delta_s, \delta_{s+a}, \rho, q, \mathcal{L}$ and κ .

Remark 2.2. 1) For $q = 1$, Liu et al. [29] considered the problem of recovering signals which are compressible in a general frame D via l_1 -analysis, with the assumption that the measurement matrix A satisfies an \tilde{D} -RIP condition, where \tilde{D} is a general dual frame of D . Note that D^\dagger is the canonical dual frame of D , and Theorem 2.1 might be extended to (\tilde{D}, q) -RIP recovery results where \tilde{D} is a general dual frame. However, since in most cases, A is a random matrix, using a general dual frame \tilde{D} would not lead to any advantage.

2) The above theorem requires that the measurement matrix A satisfies an (D^\dagger, q) -RIP condition (2.4). From the coming subsection, we will see that such a condition is met

by setting $a = O(s)$ and $m = O(s + qs \log n)$. In this case, the last term of (2.6) is roughly $C(s + qs \log n)^{1/q-1/r}$.

The proof of the above theorem involves several lemmas. We postpone the proof in Subsection 2.4. A direct consequence of the theorem is the following corollary, which is useful for the proof of Theorem 1.1

Corollary 2.3. *Under the assumptions of Theorem 2.1, we further assume that $z = 0$. Then any solution \hat{f} to (P_q) satisfies*

$$\|\hat{f} - f\|_2 \leq C_1 \frac{\|D^* f - (D^* f)_{[s]}\|_q}{s^{1/q-1/2}},$$

where C_1 is given by Theorem 2.1.

2.2 Random Gaussian measurements implying (D^\dagger, q) -RIP

We next determine how many random Gaussian observations are needed to guarantee that the (D^\dagger, q) -RIP condition in Theorem 2.1 holds with high probability. Let A be an $m \times n$ matrix whose entries are i.i.d Gaussian random variables $\mathcal{N}(0, \sigma^2)$. For a given q , let $\varrho_q = \sigma^q 2^{q/2} \Gamma(\frac{q+1}{2}) / \sqrt{\pi}$. Using a same argument as that for [12, Lemma 3.2], one can prove the following result.

Lemma 2.4. *Let $0 < q \leq 1$, A be an $m \times n$ matrix whose entries are i.i.d Gaussian random variables $\mathcal{N}(0, \sigma^2)$. Then for any fixed $x \in \mathbb{R}^n$, $\eta > 0$,*

$$\mathbb{P} \left(\left| \|Ax\|_q^q - m\varrho_q \|x\|_2^q \right| \geq \eta m\varrho_q \|x\|_2^q \right) \leq 2 \exp \left(-\frac{\eta^2 m}{2q\beta_q^2} \right), \quad (2.8)$$

where

$$\varrho_q = \sigma^q 2^{q/2} \Gamma(\frac{q+1}{2}) / \sqrt{\pi},$$

and

$$\beta_q = (31/40)^{1/4} \left[1.13 + \sqrt{q} \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}} \right)^{-1/q} \right]. \quad (2.9)$$

Remark 2.5. Note that $(\Gamma\left(\frac{q+1}{2}\right)/\sqrt{\pi})^{1/q}$ is an increasing function of q , bounded below by $e^{-\gamma/2}/2 \simeq 0.375$ [12]. Therefore

$$\beta_1 = (31/40)^{1/4}(1.13 + \sqrt{\pi}) \simeq 3.8407,$$

and when $q \rightarrow 0$, $\beta_q \rightarrow 1.13(31/40)^{1/4} \simeq 1.0602$.

Using a same argument as that for [12, Lemma 3.3], one can prove the following result Lemma 2.4. In this paper, we provide alternative simple proof for this result. Such a proof is motivated by [3].

Lemma 2.6. Let $0 < q \leq 1, \eta, \epsilon > 0$, A be an $m \times n$ matrix whose entries are i.i.d Gaussian random variables $\mathcal{N}(0, \sigma^2)$. Set $\delta = \frac{\eta + \epsilon^q}{1 - \epsilon^q}$. Then

$$(1 - \delta)m_{\mathcal{Q}_q} \|Dv\|_2^q \leq \|ADv\|_q^q \leq (1 + \delta)m_{\mathcal{Q}_q} \|Dv\|_2^q \quad (2.10)$$

holds uniformly for all k sparse vectors $v \in \mathbb{R}^d$ with probability exceeding

$$1 - 2 \left(\frac{3ed}{\epsilon k} \right)^k \exp \left(-\frac{\eta^2 m}{2q\beta_q^2} \right). \quad (2.11)$$

Proof. First note that it suffices to prove (2.10) in the case of $\|Dv\|_2 = 1$, since A is linear. Let $\Sigma_k = \{Dv : \|v\|_0 \leq k\}$. Fix an index set $T \subset [d]$ with $|T| = k$, denote by X_T the subspace spanned by the columns of D_T . Note that X_T is at most k dimensional, and we endow the l_2 norms. Choose a finite ϵ covering of the unit sphere in X_T , i.e., a set of points $Q_T \subset X_T$, with $\|u\|_2 = 1$ for all $u \in Q_T$, such that for all $v \in X_T$, $\|v\|_2 = 1$, we have

$$\min_{u \in Q_T} \|v - u\|_2 \leq \epsilon.$$

According to [32, Lemma 2.2], there exists such an Q_T with $|Q_T| \leq \left(\frac{3}{\epsilon}\right)^k$. Repeat this process for each possible index set T , and collect all the sets Q_T together:

$$Q = \bigcup_{T: |T|=k} Q_T.$$

Since the number of possible T is $\binom{d}{k}$, thus, by Sterling's approximation,

$$|Q| \leq \binom{d}{k} \left(\frac{3}{\epsilon}\right)^k \leq \left(\frac{ed}{k}\right)^k \left(\frac{3}{\epsilon}\right)^k = \left(\frac{3ed}{\epsilon k}\right)^k.$$

Applying Lemma 2.4, one gets that

$$\mathbb{P} \left(\max_{u \in Q} \left| \|Au\|_q^q - m_{\mathcal{Q}_q} \|u\|_2^q \right| \geq \eta m_{\mathcal{Q}_q} \|u\|_2^q \right) \leq 2 \left(\frac{3ed}{\epsilon k}\right)^k \exp \left(-\frac{\eta^2 m}{2q\beta_q^2} \right).$$

It thus follows that with probability exceeding (2.11),

$$m_{\mathcal{Q}_q}(1 - \eta) \|u\|_2^q \leq \|Au\|_q^q \leq m_{\mathcal{Q}_q}(1 + \eta) \|u\|_2^q \quad \text{for all } u \in Q. \quad (2.12)$$

Now define B as the smallest number such that

$$\|Av\|_q^q \leq m_{\mathcal{Q}_q}(1 + B) \|v\|_2^q \quad \text{for all } v \in \Sigma_k, \|v\|_2 = 1.$$

Our goal is to show that $B \leq \delta$. Note that from the definitions of Q_T and Q , we know that for any $v \in \Sigma_k, \|v\|_2 = 1$, we can choose an $u \in Q$ such that $\|u - v\|_2 \leq \epsilon$ and such that $u - v \in \Sigma_k$. Thus, we get

$$\|Av\|_q^q \leq \|A(v - u)\|_q^q + \|Au\|_q^q \leq m_{\mathcal{Q}_q}[(1 + B)\epsilon^q + 1 + \eta].$$

It thus follows from the definition of B that

$$m_{\mathcal{Q}_q}(1 + B) \leq m_{\mathcal{Q}_q}[(1 + B)\epsilon^q + 1 + \eta],$$

which leads to $B \leq \delta$. The lower inequality follows from this since by triangle inequality for $\|\cdot\|_q^q$,

$$\|Av\|_q^q \geq \|Au\|_q^q - \|A(u - v)\|_q^q \geq m_{\mathcal{Q}_q}[(1 - \eta) - (1 + \delta)\epsilon^q] = m_{\mathcal{Q}_q}(1 - \delta).$$

The proof is finished. □

2.3 Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. The proof is similar to [12], with a simple modification. By Corollary 2.3, we only need to prove that the (D^\dagger, q) -RIP condition (2.4) holds with high probability. Let $a = ts = \lceil (2^{q/2} b \kappa^q)^{\frac{2}{2-q}} \rceil s$ and $k = a + s = (\lceil (2^{q/2} b \kappa^q)^{\frac{2}{2-q}} \rceil + 1)s$, where $b > 1$. Note that

$$\rho^{1-q/2} (\rho^{2/q-1} + 1)^{q/2} \kappa^q < \rho^{1-q/2} 2^{q/2} \kappa^q \leq b^{-1}.$$

Therefore, a sufficient condition for (2.4) is $\delta_{(t+1)s} < \frac{b-1}{b+1}$. Choose $\eta = r(b-1)/(b+1)$ for $r \in (0, 1)$ and $\epsilon^q = (1-r)(b-1)/(2b)$. We have $(\eta + \epsilon^q)/(1 - \epsilon^q) \leq (b-1)/(b+1)$. By Lemma 2.6, A will fail to satisfy (D^\dagger, q) -RIP (2.10) with probability less than

$$2 \left(\frac{3ed}{\epsilon k} \right)^k \exp \left(-\frac{\eta^2 m}{2q\beta_q^2} \right).$$

It is enough to prove that the above quantity can be bounded by $(\frac{s}{ed})^s \leq 1/\binom{d}{s}$. This is equivalent to

$$\begin{aligned} m &\geq \frac{2q\beta_q^2}{\eta^2} \left[k \ln \left(\frac{3ed}{k} \right) + k \ln \left(\frac{1}{\epsilon} \right) + \ln 2 + s \ln \left(\frac{ed}{s} \right) \right] \\ &= \frac{2q\beta_q^2(b+1)^2}{r^2(b-1)^2} \left[\left(\lceil (2^{q/2} b \kappa^q)^{\frac{2}{2-q}} \rceil + 1 \right) s \left(\ln \left(\frac{ed}{s} \right) + \ln 3 - \ln \left(\lceil (2^{q/2} b \kappa^q)^{\frac{2}{2-q}} \rceil + 1 \right) \right) \right. \\ &\quad \left. + \frac{1}{q} (\lceil (2^{q/2} b \kappa^q)^{\frac{2}{2-q}} \rceil + 1) s \ln \left(\frac{2b}{(1-r)(b-1)} \right) + \ln 2 + s \ln \left(\frac{ed}{s} \right) \right]. \end{aligned}$$

Similar to [12], by setting $r = 0.849$ and $b = 5$,

$$\begin{aligned} m &\geq 6.25q\beta_q^2 \left[\left(\lceil (5 \cdot 2^{q/2} \kappa^q)^{\frac{2}{2-q}} \rceil + 1 \right) \left(\ln 3 - \ln \left(\lceil (5 \cdot 2^{q/2} \kappa^q)^{\frac{2}{2-q}} \rceil + 1 \right) \right) s + \ln 2 \right. \\ &\quad \left. + \left(\lceil (5 \cdot 2^{q/2} \kappa^q)^{\frac{2}{2-q}} \rceil + 2 \right) s \ln \left(\frac{ed}{s} \right) \right] + 17.6\beta_q^2 (\lceil (5 \cdot 2^{q/2} \kappa^q)^{\frac{2}{2-q}} \rceil + 1)s. \end{aligned}$$

The proof is finished.

2.4 Proof of Theorem 2.1

In this subsection, we prove Theorem 2.1. Our goal is to bound the norm of $h = \hat{f} - f$, where \hat{f} and f are as in the theorem.

We begin by establishing several lemmas for a general vector h . For arbitrary fixed $h \in \mathbb{R}^n$, since D is a frame for \mathbb{R}^n , one can upper bound $\|h\|_2^2$ by $\mathcal{L}^{-1}\|D^*h\|_2^2$. To estimate $\|D^*h\|_2$, we use a common decomposition technique in the standard compressed sensing (e.g., [9]). We write $D^*h = (x_1, \dots, x_d)^T$. Rearranging the indices if necessary, we may assume that $|x_1| \geq |x_{s+2}| \geq \dots \geq |x_d|$. Set $T = T_0 = \{1, 2, \dots, s\}$, $T_1 = \{s+1, s+2, \dots, s+a\}$, and $T_i = \{s+(i-1)a+1, \dots, s+ia\}$, $i = 2, \dots$, with the last subset of size less than or equal to a . Denote $T_{01} = T_0 \cup T_1$. Note that by applying the first inequality of (2.1), we have

$$\mathcal{L}^{q/2}\|h\|_2^q \leq \|D^*h\|_2^q = \left(\|D_{T_{01}}^*h\|_2^2 + \sum_{j \geq 2} \|D_{T_j}^*h\|_2^2 \right)^{q/2} \leq \|D_{T_{01}}^*h\|_2^q + \sum_{j \geq 2} \|D_{T_j}^*h\|_2^q. \quad (2.13)$$

In what follows, we shall upper bound the last two terms. The following lemma, which was originally proved in [12], gives an upper bound of the tail $\sum_{j \geq 2} \|D_{T_j}^*h\|_2^q$ in terms on $\|D_{T^c}^*h\|_q^q$. We prove it for completeness.

Lemma 2.7 (Bounding the tail). *We have*

$$\sum_{j \geq 2} \|D_{T_j}^*h\|_2^q \leq a^{q/2-1} \|D_{T^c}^*h\|_q^q. \quad (2.14)$$

Proof. Fix $i > 0$, for each $l \in T_i$ and $l' \in T_{i+1}$, obviously we have $|x_{l'}|^q \leq |x_l|^q$. Thus, $|x_{l'}|^q \leq \|D_{T_i}^*h\|_q^q/a$. It thus follows that

$$\sum_{j \geq 2} \|D_{T_j}^*h\|_2^q \leq a^{q/2-1} \sum_{j \geq 1} \|D_{T_j}^*h\|_q^q = a^{q/2-1} \|D_{T^c}^*h\|_q^q.$$

□

To bound $\|D_{T_{01}}^*h\|_2^q$, we need the following result which utilizes the fact that A satisfies the (D^\dagger, q) -RIP.

Lemma 2.8 (Consequence of the (D^\dagger, q) -RIP). *Assume that A satisfies the (D^\dagger, q) -RIP of order $s + a$. Let*

$$\Delta = \frac{1 + \delta_a}{1 - \delta_{s+a}}. \quad (2.15)$$

Then, we have

$$\|DD_{T_{01}}^* h\|_2^2 \leq \kappa \mathcal{U} \Delta^{2/q} a^{1-2/q} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{1 + \delta_a} \right)^{2/q}. \quad (2.16)$$

Proof. Note that $D^\dagger D^* = I$, thus,

$$\begin{aligned} \|Ah\|_q^q &= \|AD^\dagger D^* h\|_q^q = \left\| AD^\dagger D_{T_{01}}^* h + \sum_{j \geq 2} AD^\dagger D_{T_j}^* h \right\|_q^q \\ &\geq \|AD^\dagger D_{T_{01}}^* h\|_q^q - \sum_{j \geq 2} \|AD^\dagger D_{T_j}^* h\|_q^q, \end{aligned}$$

where the last inequality follows from the triangle inequality (2.2). It thus follows from the definitions of (D^\dagger, q) -RIP that

$$\|Ah\|_q^q \geq (1 - \delta_{s+a}) \|D^\dagger D_{T_{01}}^* h\|_2^q - (1 + \delta_a) \sum_{j \geq 2} \|D^\dagger D_{T_j}^* h\|_2^q.$$

Using the definition of frame (1.1), which is equivalent to

$$\mathcal{L} \leq \lambda_{\min}(DD^*) \leq \lambda_{\max}(DD^*) \leq \mathcal{U}, \quad (2.17)$$

and implies

$$\mathcal{U}^{-1} \leq \lambda_{\min}((DD^*)^{-1}) \leq \lambda_{\max}((DD^*)^{-1}) \leq \mathcal{L}^{-1},$$

and recalling that $D^\dagger = (DD^*)^{-1}D$, we get

$$\|Ah\|_q^q \geq \mathcal{U}^{-q} (1 - \delta_{s+a}) \|DD_{T_{01}}^* h\|_2^q - \mathcal{L}^{-\frac{q}{2}} (1 + \delta_a) \sum_{j \geq 2} \|D_{T_j}^* h\|_2^q.$$

Introducing (2.14) to the above,

$$\|Ah\|_q^q \geq \mathcal{U}^{-q} (1 - \delta_{s+a}) \|DD_{T_{01}}^* h\|_2^q - \mathcal{L}^{-\frac{q}{2}} (1 + \delta_a) a^{q/2-1} \|D_{T^c}^* h\|_q^q.$$

Rearranging terms, noting δ and κ are given by (2.15) and (2.5) respectively, we get that

$$\begin{aligned} \|DD_{T_{01}}^* h\|_2^q &\leq \Delta \kappa^{q/2} \mathcal{U}^{q/2} a^{q/2-1} \|D_{T^c}^* h\|_q^q + \mathcal{U}^q \|Ah\|_q^q / (1 - \delta_{s+a}) \\ &= \Delta \kappa^{q/2} \mathcal{U}^{q/2} a^{q/2-1} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{1 + \delta_a} \right). \end{aligned}$$

Taking the $(2/q)$ -th power on both sides, we get the desired result. \square

With the estimation on $\|DD_{T_{01}}^* h\|_2^2$, we are ready to give an upper bound on $\|D_{T_{01}}^* h\|_2^q$. This can be done by developing a relationship between $\|DD_{T_{01}}^* h\|_2^2$ and $\|D_{T_{01}}^* h\|_2^2$.

Lemma 2.9 (Bounding $\|D_{T_{01}}^* h\|_2^q$). *Under the assumptions and notations of Lemma 2.8, we have*

$$\|D_{T_{01}}^* h\|_2^q \leq 2^{-q/2} \left(1 + \sqrt{1 + 4\kappa^{-2} \Delta^{-2/q}} \right)^{q/2} \kappa^q \Delta a^{q/2-1} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{1 + \delta_a} \right). \quad (2.18)$$

Proof. Note that by Cauchy-Schwarz inequality and (1.1)

$$\|D_{T_{01}}^* h\|_2^4 = |\langle h, DD_{T_{01}}^* h \rangle|^2 \leq \|h\|_2^2 \|DD_{T_{01}}^* h\|_2^2 \leq \mathcal{L}^{-1} \|D^* h\|_2^2 \|DD_{T_{01}}^* h\|_2^2.$$

Substituting with $\|D^* h\|_2^2 = \|D_{T_{01}}^* h\|_2^2 + \sum_{j \geq 2} \|D_{T_j}^* h\|_2^2$, and then applying the first inequality of (2.1) to upper bound the term $\sum_{j \geq 2} \|D_{T_j}^* h\|_2^2$, we get

$$\|D_{T_{01}}^* h\|_2^4 \leq \mathcal{L}^{-1} \|DD_{T_{01}}^* h\|_2^2 \left(\|D_{T_{01}}^* h\|_2^2 + \left(\sum_{j \geq 2} \|D_{T_j}^* h\|_2^q \right)^{2/q} \right). \quad (2.19)$$

Introducing (2.14) to the above,

$$\|D_{T_{01}}^* h\|_2^4 \leq \mathcal{L}^{-1} \|DD_{T_{01}}^* h\|_2^2 \left(\|D_{T_{01}}^* h\|_2^2 + a^{1-2/q} \|D_{T^c}^* h\|_q^2 \right).$$

Rearranging terms and completing the square, this reads as

$$\left(\|D_{T_{01}}^* h\|_2^2 - \frac{\|DD_{T_{01}}^* h\|_2^2}{2\mathcal{L}} \right)^2 \leq \|DD_{T_{01}}^* h\|_2^2 \left(\frac{\|DD_{T_{01}}^* h\|_2^2}{4\mathcal{L}^2} + \frac{\|D_{T^c}^* h\|_q^2}{\mathcal{L} a^{2/q-1}} \right).$$

Taking square root of each side and rearranging terms, we get

$$\|D_{T_{01}}^* h\|_2^2 \leq \frac{\|DD_{T_{01}}^* h\|_2^2}{2\mathcal{L}} + \sqrt{\|DD_{T_{01}}^* h\|_2^2 \left(\frac{\|DD_{T_{01}}^* h\|_2^2}{4\mathcal{L}^2} + \frac{\|D_{T^c}^* h\|_q^2}{\mathcal{L}a^{2/q-1}} \right)}.$$

Recalling δ and κ are given by (2.15) and (2.5) respectively, and upper bounding the term $\frac{\|D_{T^c}^* h\|_q^2}{\mathcal{L}a^{2/q-1}}$ by

$$\frac{a^{1-2/q}(\|D_{T^c}^* h\|_q^q)^{2/q}}{\mathcal{L}} \leq \frac{\kappa\mathcal{U}\Delta^{2/q}a^{1-2/q} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{1+\delta_a} \right)^{2/q}}{\kappa\mathcal{U}\mathcal{L}\Delta^{2/q}},$$

and applying (2.16), we have

$$\begin{aligned} \|D_{T_{01}}^* h\|_2^2 &\leq \left(\frac{1}{2\mathcal{L}} + \sqrt{\frac{1}{4\mathcal{L}^2} + \frac{1}{\kappa\mathcal{U}\mathcal{L}\Delta^{2/q}}} \right) \kappa\mathcal{U}\Delta^{2/q}a^{1-2/q} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{1+\delta_a} \right)^{2/q} \\ &= 2^{-1} \left(1 + \sqrt{1 + 4\kappa^{-2}\Delta^{-2/q}} \right) \kappa^2\Delta^{2/q}a^{1-2/q} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{1+\delta_a} \right)^{2/q}. \end{aligned}$$

Taking the $(q/2)$ -th power on both sides, we get our desired result. \square

By lemmas 2.7 and 2.9, we know that the last two terms of (2.13) can be upper bounded in terms of $\|D_{T^c}^* h\|_q^q$ and $\|Ah\|_q^q$. In what follows, we develop another two lemmas to bound $\|D_{T^c}^* h\|_q^q$ and $\|Ah\|_q^q$. The following lemma shows that an suitable (D^\dagger, q) -RIP condition implies the robust D -NSP $_q$ of the measurement matrix A .

Lemma 2.10 (Robust D -NSP $_q$). *Under the assumptions and notations of Lemma 2.8, we have for any $h \in \mathbb{R}^N$,*

$$\|D_T^* h\|_q^q \leq \theta \|D_{T^c}^* h\|_q^q + \frac{\theta\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{1+\delta_a}, \quad (2.20)$$

where

$$\theta = 2^{-q/2} \left(1 + \sqrt{1 + 4\kappa^{-2}\Delta^{-2/q}} \right)^{q/2} \kappa^q \Delta \rho^{1-q/2}, \quad (2.21)$$

where ρ is given by (2.5). In particular, if the condition (2.4) is satisfied, then $\theta < 1$.

Proof. Using $\|D_{T_0}^* h\|_2 \geq \|D_{T_0}^* h\|_2$ and then applying Hölder's inequality mentioned in (2.1) to lower bound $\|D_{T_0}^* h\|_2$, we get $\|D_{T_0}^* h\|_2^q \geq s^{q/2-1} \|D_T^* h\|_q^q$. Therefore, combining with (2.18), we get

$$s^{q/2-1} \|D_T^* h\|_q^q \leq 2^{-q/2} \left(1 + \sqrt{1 + 4\kappa^{-2} \Delta^{-2/q}}\right)^{q/2} \kappa^q \Delta a^{q/2-1} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{1 + \delta_a} \right).$$

Dividing both sides by $s^{1-2/q}$, we get (2.20). It remains to show that $\theta < 1$. Actually, this can be verified by showing that $\theta^{2/q} < 1$, that is

$$2^{-1} \left(1 + \sqrt{1 + 4\kappa^{-2} \Delta^{-2/q}}\right) \kappa^2 \Delta^{2/q} \rho^{2/q-1} < 1.$$

Multiplying both sides by $\sqrt{1 + 4\kappa^{-2} \Delta^{-2/q}} - 1$, this reads as

$$2^{-1} \cdot 4\kappa^{-2} \Delta^{-2/q} \cdot \kappa^2 \Delta^{2/q} \rho^{2/q-1} < \sqrt{1 + 4\kappa^{-2} \Delta^{-2/q}} - 1,$$

which is equivalent to

$$2\rho^{2/q-1} + 1 < \sqrt{1 + 4\Delta^{-2/q} \kappa^{-2}}.$$

Taking the second power of both sides, subtracting both sides by 1 and by a simple calculation, this reads as

$$4\rho^{2/q-1}(\rho^{2/q-1} + 1) < 4\Delta^{-2/q} \kappa^{-2}.$$

Dividing both sides by $4\Delta^{-2/q} \kappa^{-2}$, and then taking the $(q/2)$ -th power on both sides, we know that this is equivalent to

$$\rho^{1-q/2}(\rho^{2/q-1} + 1)^{q/2} \Delta \kappa^q < 1.$$

Introducing (2.15), we know that the above inequality is equivalent to (2.4). Consequently, (2.4) implies $\theta < 1$. \square

Note that the above lemmas hold for any $h \in \mathbb{R}^n$. In what follows, we shall choose $h = \hat{f} - f$, where \hat{f} is a solution of (2.3) and f is the original signal. Let Ω be the index set

of the largest s entries of D^*f in magnitude. The following results can be verified by using the fact that \hat{f} is a solution of (2.3).

Lemma 2.11 (Consequence of a solution). *Let $h = \hat{f} - f$, where \hat{f} is a solution of (2.3) and f satisfies $\|Af - y\|_r \leq \varepsilon$. Let Ω be the index set of the largest s entries of D^*f in magnitude. Then we have*

$$\|Ah\|_q^q \leq m^{1-q/r}(2\varepsilon)^q \quad (2.22)$$

and

$$\|D_{T^c}^*h\|_q^q \leq \|D_T^*h\|_q^q + 2\|D_{\Omega^c}^*f\|_q^q. \quad (2.23)$$

Proof. Since both \hat{f} and f are feasible and $r \geq 1$, we have

$$\|Ah\|_r \leq \|Af - y\|_r + \|A\hat{f} - y\|_r \leq 2\varepsilon.$$

Using the Hölder's inequality mentioned in (2.1), we get

$$\|Ah\|_q^q \leq m^{1-q/r}\|Ah\|_r^q \leq m^{1-q/r}(2\varepsilon)^q.$$

This proves (2.22).

Since \hat{f} is a minimizer of (2.3), one gets that

$$\|D^*f\|_q^q \geq \|D^*\hat{f}\|_q^q.$$

That is

$$\|D_{\Omega}^*f\|_q^q + \|D_{\Omega^c}^*f\|_q^q \geq \|D_{\Omega}^*\hat{f}\|_q^q + \|D_{\Omega^c}^*\hat{f}\|_q^q.$$

Substituting with $\hat{f} = f + h$ and using the triangle inequality (2.2),

$$\|D_{\Omega}^*f\|_q^q + \|D_{\Omega^c}^*f\|_q^q \geq \|D_{\Omega}^*f\|_q^q - \|D_{\Omega}^*h\|_q^q + \|D_{\Omega^c}^*h\|_q^q - \|D_{\Omega^c}^*f\|_q^q.$$

Rearranging terms,

$$\|D_{\Omega^c}^*h\|_q^q - \|D_{\Omega}^*h\|_q^q \leq 2\|D_{\Omega^c}^*f\|_q^q.$$

Combining with the fact that

$$\|D_{T^c}^* h\|_q^q - \|D_T^* h\|_q^q \leq \|D_{\Omega^c}^* h\|_q^q - \|D_{\Omega}^* h\|_q^q,$$

and rearranging terms, we get (2.23). The proof is finished. \square

We may now conclude the proof of Theorem 2.1. We first apply lemmas 2.10 and 2.11 to get an upper bound on $\|D_{T^c}^* h\|_q^q$. Introducing (2.20) to (2.23), we get

$$\|D_{T^c}^* h\|_q^q \leq \|D_T^* h\|_q^q + 2\|D_{\Omega^c}^* f\|_q^q \leq \theta\|D_{T^c}^* h\|_q^q + \frac{\theta\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{1+\delta_a} + 2\|D_{\Omega^c}^* f\|_q^q.$$

Rearranging terms and dividing both sides by $1-\theta$ (noting that $\theta < 1$ by Lemma 2.10), we get

$$\|D_{T^c}^* h\|_q^q \leq \frac{\theta\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{(1-\theta)(1+\delta_a)} + \frac{2}{1-\theta}\|D_{\Omega^c}^* f\|_q^q. \quad (2.24)$$

Now we can upper bound $\|h\|_2$. Introducing (2.14) and (2.18) to (2.13), and noting that θ and ρ are given by (2.21) and (2.5) respectively, we get

$$\begin{aligned} \mathcal{L}^{q/2}\|h\|_2^q &\leq \theta s^{q/2-1} \left(\|D_{T^c}^* h\|_q^q + \frac{\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{1+\delta_a} \right) + a^{q/2-1}\|D_{T^c}^* h\|_q^q \\ &= \frac{(\theta + \rho^{1-q/2})\|D_{T^c}^* h\|_q^q}{s^{1-q/2}} + \frac{\theta\mathcal{L}^{q/2}\rho^{q/2-1}\|Ah\|_q^q}{1+\delta_a}. \end{aligned}$$

Applying (2.24) to the above, we get

$$\begin{aligned} \mathcal{L}^{q/2}\|h\|_2^q &\leq \frac{(\theta + \rho^{1-q/2})}{s^{1-q/2}} \left(\frac{\theta\mathcal{L}^{q/2}a^{1-q/2}\|Ah\|_q^q}{(1-\theta)(1+\delta_a)} + \frac{2}{1-\theta}\|D_{\Omega^c}^* f\|_q^q \right) + \frac{\theta\mathcal{L}^{q/2}\rho^{q/2-1}\|Ah\|_q^q}{1+\delta_a} \\ &= \frac{2(\theta + \rho^{1-q/2})\|D_{\Omega^c}^* f\|_q^q}{(1-\theta)s^{1-q/2}} + \frac{\theta\mathcal{L}^{q/2}(1+\rho^{q/2-1})\|Ah\|_q^q}{(1-\theta)(1+\delta_a)}. \end{aligned}$$

Using (2.22) to the above, and dividing both sides by $\mathcal{L}^{q/2}$,

$$\|h\|_2^q \leq \frac{2(\theta + \rho^{1-q/2})\|D_{\Omega^c}^* f\|_q^q}{\mathcal{L}^{q/2}(1-\theta)s^{1-q/2}} + \frac{\theta(1+\rho^{q/2-1})m^{1-q/r}(2\varepsilon)^q}{(1-\theta)(1+\delta_a)}.$$

Taking the $(1/q)$ -th power on both sides and then using a basic inequality $(b + c)^{1/q} \leq 2^{1/q-1}(b^{1/q} + c^{1/q})$, $\forall b, c \geq 0$ we get

$$\|h\|_2 \leq \frac{(2\theta + 2\rho^{1-q/2})^{1/q} \|D_{\Omega^c}^* f\|_q}{\sqrt{\mathcal{L}}(1-\theta)^{1/q} s^{1/q-1/2}} + \frac{(2\theta + 2\theta\rho^{q/2-1})^{1/q} m^{1/q-1/r} \varepsilon}{(1-\theta)^{1/q} (1+\delta_a)^{1/q}}.$$

Thus, we get (2.6). It remains to prove (2.7). By (2.20),

$$\|D^* h\|_q^q = \|D_T^* h\|_q^q + \|D_{T^c}^* h\|_q^q \leq (1+\theta) \|D_{T^c}^* h\|_q^q + \frac{\theta \mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{1+\delta_a}.$$

Introducing (2.24) and then using (2.22), we get

$$\begin{aligned} \|D^* h\|_q^q &\leq (1+\theta) \left(\frac{\theta \mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{(1-\theta)(1+\delta_a)} + \frac{2}{1-\theta} \|D_{\Omega^c}^* f\|_q^q \right) + \frac{\theta \mathcal{L}^{q/2} a^{1-q/2} \|Ah\|_q^q}{1+\delta_a} \\ &\leq \frac{2(1+\theta)}{1-\theta} \|D_{\Omega^c}^* f\|_q^q + \frac{2^{q+1} \theta \mathcal{L}^{q/2} a^{1-q/2} m^{1-q/r} \varepsilon^q}{(1-\theta)(1+\delta_a)}. \end{aligned}$$

Thus, we get the desired result (2.7). The proof is finished.

Remark 2.12. In the proof, we derive an upper bound for $\|D^* \hat{f} - D^* f\|_2$:

$$\|D^* \hat{f} - D^* f\|_2 \leq C_5 \frac{\|D^* f - (D^* f)_{[s]}\|_q}{s^{1/q-1/2}} + C_6 m^{1/q-1/r} \varepsilon.$$

3 Compressed data separation via l_q split analysis

In this section, we prove Theorem 1.5. The proof is similar to that of Theorem 1.1, and makes use some ideas from [28]. We first establish an (D, q) -RIP recovery result for compressed data separation with ι components ($\iota \geq 2$). Considering $\iota = 2$, we then utilize Lemma 2.6 to show that such an (D, q) -RIP condition holds with high probability. As a result, one can finish the proof.

Let ι be a positive integer greater than 2 and $D_1 \in \mathbb{R}^{n \times d_1}, D_2 \in \mathbb{R}^{n \times d_2}, \dots, D_\iota \in \mathbb{R}^{n \times d_\iota}$ be ι tight frames with frames bounds 1 for \mathbb{R}^n . Set $s = s_1 + s_2 + \dots + s_\iota$ and $\bar{d} = d_1 + d_2 + \dots + d_\iota$.

Let

$$\bar{D} = [D_1 | D_2 | \cdots | D_\iota], \Psi = \begin{pmatrix} D_1 & & \\ & D_2 & \\ & & \ddots \\ & & & D_\iota \end{pmatrix}, \text{ and } f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_\iota \end{pmatrix}. \quad (3.1)$$

Note that from the definition of tight frames,

$$\sum_{j=1}^{\iota} f_j = \sum_{j=1}^{\iota} D_j D_j^* f_j = \bar{D} \Psi^* f.$$

Then, with $\iota = 2$, (1.5) can be rewritten as

$$\hat{f} = \arg \min_{\tilde{f} \in \mathbb{R}^{\iota n}} \|\Psi^* \tilde{f}\|_q^q \quad \text{s.t.} \quad A \bar{D} \Psi^* \tilde{f} = y. \quad (3.2)$$

We define the mutual coherence between D_1, D_2, \dots, D_ι as follows.

Definition 3.1. Let $D_1 = (d_{1i})_{1 \leq i \leq d_1}, D_2 = (d_{2j})_{1 \leq j \leq d_2}, \dots, D_\iota = (d_{\iota l})_{1 \leq l \leq d_\iota}$. The mutual coherence between D_1, D_2, \dots, D_ι is defined as

$$\mu_1 = \mu_1(D_1; D_2; \dots; D_\iota) = \max_{k \neq l} \max_{i, j} |\langle d_{ki}, d_{lj} \rangle|.$$

3.1 (\bar{D}, q) -RIP recovery result for l_q split analysis

In this subsection, we give (\bar{D}, q) -RIP guarantee results on compressed data separation from noisy measurements $y = A(f_1 + f_2 + \cdots + f_\iota) + z$ via solving the following l_q -analysis optimization

$$\arg \min_{\tilde{f} \in \mathbb{R}^{\iota n}} \|\Psi^* \tilde{f}\|_q \quad \text{subject to} \quad \|A \bar{D} \Psi^* \tilde{f} - y\|_r \leq \varepsilon, \quad (3.3)$$

where $0 < q \leq 1 \leq r \leq \infty, \varepsilon \geq 0, \|z\|_r \leq \varepsilon$ and \bar{D}, Ψ, f are as in (3.1).

Theorem 3.2. Let $\iota \in \mathbb{Z}^+, 0 < q \leq 1 \leq r \leq \infty, \varepsilon \geq 0$. Suppose we observe data from the model $y = A(f_1 + f_2 + \cdots + f_\iota) + z$ with $\|z\|_r \leq \varepsilon$. Let $D_1 \in \mathbb{R}^{n \times d_1}, D_2 \in \mathbb{R}^{n \times d_2}, \dots, D_\iota \in \mathbb{R}^{n \times d_\iota}$,

be ι arbitrary tight frames for \mathbb{R}^n with frame bounds 1. Let \bar{D}, Ψ, f be as in (3.1). Fix a positive integer $a > s$. Assume that the mutual coherence μ_1 between D_1, D_2, \dots, D_ι satisfies

$$\mu_1(s+a)(\rho^{2/q-1} + 1) < 1, \quad (3.4)$$

and that the (\bar{D}, q) -RIP constant of A satisfies

$$\Delta \rho^{1-q/2}(\rho^{2/q-1} + 1)^{q/2} < (2\iota)^{-q/2}, \quad (3.5)$$

where

$$\rho = \frac{s}{a} \quad \text{and} \quad \Delta = \frac{1 + \delta_a}{1 - \delta_{s+a}}. \quad (3.6)$$

Then any solution \hat{f} to the l_q Split analysis (3.3) obeys

$$\|\hat{f} - f\|_2 \leq \tilde{C}_1 \frac{\|\Psi^* f - (\Psi^* f)_{[s]}\|_q}{s^{1/q-1/2}} + \tilde{C}_2 m^{1/q-1/r} \varepsilon, \quad (3.7)$$

and

$$\|\Psi^* \hat{f} - \Psi^* f\|_q^q \leq \tilde{C}_3 \|\Psi^* f - (\Psi^* f)_{[s]}\|_q^q + \tilde{C}_4 a^{1-q/2} m^{1-q/r} \varepsilon^q, \quad (3.8)$$

where $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3$ and \tilde{C}_4 are positive constants (given explicitly in the proof) depending only on the (\bar{D}, q) -RIP constant $\delta_s, \delta_{s+a}, \rho, \iota$ and q .

The proof of this theorem will be given in Subsection 3.3. The following result is a direct consequence of the above theorem. We will use it to prove Theorem 1.5.

Corollary 3.3. *Under the assumptions of Theorem 3.2, we further assume that $z = 0$. Then any solution \hat{f} to (3.2) satisfies*

$$\|\hat{f} - f\|_2 \leq \tilde{C}_1 \frac{\|\Psi^* f - (\Psi^* f)_{[s]}\|_q}{s^{1/q-1/2}},$$

where \tilde{C}_1 is given by Theorem 3.2.

3.2 Proof of Theorem 1.5

Now, we are ready to prove Theorem 1.5. The rest of the proof is similar to the argument used in the proof of Theorem 1.1. We include the sketch only. We first prove that (3.5) holds with high probability. Setting $\iota = 2$ in Theorem 3.2, it suffices to prove

$$\Delta\rho^{1-q/2} < 2^{-3q/2}.$$

Let $k = a + s = (t + 1)s = (\lceil (2^{3q/2}b)^{\frac{2}{2-q}} \rceil + 1)s$, where $b > 1$. We only need to prove that $\delta_{(t+1)s} < \frac{b-1}{b+1}$. A similar argument as that in the proof of Theorem 1.1, one can easily prove that $\delta_{(t+1)s} < \frac{b-1}{b+1}$ is met with probability exceeding $1/\binom{d}{s}$ provided that

$$\begin{aligned} m \geq & 6.25q\beta_q^2 \left[\left(\lceil (5 \cdot 2^{3q/2})^{\frac{2}{2-q}} \rceil + 1 \right) \left(\ln 3 - \ln \left(\lceil (5 \cdot 2^{3q/2})^{\frac{2}{2-q}} \rceil + 1 \right) \right) s + \ln 2 \right. \\ & \left. + \left(\lceil (5 \cdot 2^{3q/2})^{\frac{2}{2-q}} \rceil + 2 \right) s \ln \left(\frac{ed}{s} \right) \right] + 17.6\beta_q^2 (\lceil (5 \cdot 2^{3q/2})^{\frac{2}{2-q}} \rceil + 1)s. \end{aligned}$$

Note that in the proof we set $b = 5$. In this case, (3.4) is implied by

$$\mu_1 s \left(\lceil (2^{3q/2}5)^{\frac{2}{2-q}} \rceil + 1 \right) \left(\frac{1}{8 \cdot 5^{2/q}} + 1 \right) < 1.$$

Now one can finish the proof by applying Corollary 3.3 and

$$\|\Psi^* f - (\Psi^* f)_{[s]}\|_q^q \leq \|D_1^* f_1 - (D_1^* f_1)_{[s_1]}\|_q^q + \|D_2^* f_2 - (D_2^* f_2)_{[s_2]}\|_q^q.$$

3.3 Proof of Theorem 3.2

In this subsection, we prove Theorem 3.2. Our goal is to bound the norm of h , where $h = f - \hat{f}$, \hat{f} is a solution of (3.2) and f is the original signal. As in the proof of Theorem 2.1, We do so by bounding the norm of $\Psi^* h$, since Ψ is a tight frame for \mathbb{R}^n . Actually, since D_1, D_2, \dots, D_ℓ are tight frames with frame bounds 1, one has that

$$\|\Psi^* \tilde{f}\|_2^2 = \sum_{k=1}^{\ell} \|D_k^* \tilde{f}_k\|_2^2 = \sum_{k=1}^{\ell} \|\tilde{f}_k\|_2^2 = \|\tilde{f}\|_2^2 \quad \text{for all } \tilde{f} \in \mathbb{R}^n, \quad (3.9)$$

and that

$$\|\bar{D}\| = \sqrt{\lambda_{\max}(\bar{D}\bar{D}^*)} = \sqrt{\lambda_{\max}(\iota I)} = \sqrt{\iota}. \quad (3.10)$$

For arbitrary fixed $h \in \mathbb{R}^n$, we write $\Psi^*h = (x_1, x_2, \dots, x_{\bar{d}})^T$. Making rearrangements if necessary, we assume that $|x_1| \geq |x_2| \geq \dots |x_{\bar{d}}|$. Set $T = T_0 = \{1, 2, \dots, s\}$, $T_1 = \{s+1, s+2, \dots, s+a\}$, and $T_i = \{s+(i-1)a+1, \dots, s+ia\}$, $i = 2, \dots$, with the last subset of size less than or equal to a . Denote $T_{01} = T_0 \cup T_1$. Note that by applying the first inequality of (2.1), we have

$$\|h\|_2^2 = \|\Psi^*h\|_2^2 \leq \|\Psi_{T_{01}}^*h\|_2^2 + \left(\sum_{j \geq 2} \|\Psi_{T_j}^*h\|_2^q \right)^{2/q}. \quad (3.11)$$

We only need to upper bound the last two terms. By Lemma 2.7, we also have

$$\sum_{j \geq 2} \|\Psi_{T_j}^*h\|_2^q = a^{q/2-1} \|\Psi_{T^c}^*h\|_q^q. \quad (3.12)$$

Applying (3.12) to (3.11), we can upper bound $\|h\|_2^2$ by

$$\|h\|_2^2 \leq \|\Psi_{T_{01}}^*h\|_2^2 + a^{1-2/q} \|\Psi_{T^c}^*h\|_q^2. \quad (3.13)$$

To bound $\|\Psi_{T_{01}}^*h\|_2^q$, we need the following lemma, which gives an upper bound on $\sum_{k=1}^{\iota} \|D_k D_{kT_{01}^k}^* h_1\|_2^2$.

It can be proved by using of the definitions of the (\bar{D}, q) -RIP and the mutual coherence. Denote that $T_{01}^1 = T_{01} \cap [d_1]$ and $T_{01}^k = \{j - \sum_{l=1}^{k-1} d_l : j \in T_{01} \cap [\sum_{l=1}^{k-1} d_l + 1, \sum_{l=1}^k d_l]\}$ for any integer $k \in [2, \iota]$.

Lemma 3.4 (Consequence of the (\bar{D}, q) -RIP and the mutual coherence). *Assume that A satisfies the (\bar{D}, q) -RIP of order $s+a$. Let Δ be as in 3.6 and*

$$U = \frac{\mu_1(s+a)}{2}, \quad (3.14)$$

where u_1 is the mutual coherence between $D_1, D_2, \dots, D_{\iota}$. Then we have

$$\sum_{k=1}^{\iota} \|D_k D_{kT_{01}^k}^* h_k\|_2^2 \leq \iota \Delta^{2/q} a^{1-2/q} \left(\|\Psi_{T^c}^*h\|_q^q + \frac{a^{1-q/2} \|A\bar{D}\Psi^*h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right)^{2/q} + U \|\Psi_{T_{01}}^*h\|_2^2. \quad (3.15)$$

Proof. We first note that from the definition of μ_1 ,

$$\begin{aligned} \left| \sum_{k=1}^{\ell} \sum_{l \neq k} \langle D_k D_{kT_{01}^k}^* h_k, D_l D_{lT_{01}^l}^* h_l \rangle \right| &= 2 \left| \sum_{k=1}^{\ell} \sum_{l \neq k} \sum_{i \in T_{01}^k} \sum_{j \in T_{01}^l} \langle d_{ki}, d_{lj} \rangle \langle d_{ki}, h_k \rangle \langle h_l, d_{lj} \rangle \right| \\ &\leq \mu_1 \sum_{k=1}^{\ell} \sum_{l \neq k} \sum_{i \in T_{01}^k} \sum_{j \in T_{01}^l} |\langle d_{ki}, h_k \rangle \langle h_l, d_{lj} \rangle| = \mu_1 \sum_{k=1}^{\ell} \sum_{l \neq k} \|D_{kT_{01}^k}^* h_k\|_1 \|D_{lT_{01}^l}^* h_l\|_1. \end{aligned}$$

Using Cauchy-Schwarz inequality, for $k, l \in [\ell]$, $k \neq l$, we have

$$\begin{aligned} \|D_{kT_{01}^k}^* h_k\|_1 \|D_{lT_{01}^l}^* h_l\|_1 &\leq \sqrt{|T_{01}^k| \cdot |T_{01}^l|} \|D_{kT_{01}^k}^* h_k\|_2 \|D_{lT_{01}^l}^* h_l\|_2 \\ &\leq \frac{|T_{01}^k| + |T_{01}^l|}{2} \|D_{kT_{01}^k}^* h_k\|_2 \|D_{lT_{01}^l}^* h_l\|_2 \leq \frac{s+a}{2} \|D_{kT_{01}^k}^* h_k\|_2 \|D_{lT_{01}^l}^* h_l\|_2. \end{aligned}$$

Therefore, recalling that U is given by (3.14), we get

$$\begin{aligned} \left| \sum_{k=1}^{\ell} \sum_{l \neq k} \langle D_k D_{kT_{01}^k}^* h_k, D_l D_{lT_{01}^l}^* h_l \rangle \right| &\leq U \sum_{k=1}^{\ell} \sum_{l \neq k} \|D_{kT_{01}^k}^* h_k\|_2 \|D_{lT_{01}^l}^* h_l\|_2 \\ &\leq U \sum_{k=1}^{\ell} \|D_{kT_{01}^k}^* h_k\|_2^2 = U \|\Psi_{T_{01}}^* h\|_2^2. \end{aligned}$$

Thus, we get

$$\begin{aligned} \|\bar{D}\Psi_{T_{01}}^* h\|_2^2 &= \left\| \sum_{k=1}^{\ell} D_k D_{kT_{01}^k}^* h_k \right\|_2^2 \\ &= \sum_{k=1}^{\ell} \|D_k D_{kT_{01}^k}^* h_k\|_2^2 + \sum_{k=1}^{\ell} \sum_{l \neq k} \langle D_k D_{kT_{01}^k}^* h_k, D_l D_{lT_{01}^l}^* h_l \rangle \\ &\geq \sum_{k=1}^{\ell} \|D_k D_{kT_{01}^k}^* h_k\|_2^2 - U \|\Psi_{T_{01}}^* h\|_2^2. \end{aligned} \tag{3.16}$$

We next upper bound $\|\bar{D}\Psi_{T_{01}}^* h\|_2^2$. We do so by using properties of (\bar{D}, q) -RIP. Note that

$$\|A\bar{D}\Psi^* h\|_q^q = \left\| A\bar{D}\Psi_{T_{01}}^* h + \sum_{j \geq 2} A\bar{D}\Psi_{T_j}^* h \right\|_q^q \geq \|A\bar{D}\Psi_{T_{01}}^* h\|_q^q - \sum_{j \geq 2} \|A\bar{D}\Psi_{T_j}^* h\|_q^q.$$

According to the definition of (\bar{D}, q) -RIP, and then applying (3.10),

$$\begin{aligned} \|A\bar{D}\Psi^* h\|_q^q &\geq (1 - \delta_{s+a}) \|\bar{D}\Psi_{T_{01}}^* h\|_2^q - (1 + \delta_a) \sum_{j \geq 2} \|\bar{D}\Psi_{T_j}^* h\|_2^q \\ &\geq (1 - \delta_{s+a}) \|\bar{D}\Psi_{T_{01}}^* h\|_2^q - \ell^{q/2} (1 + \delta_a) \sum_{j \geq 2} \|\Psi_{T_j}^* h\|_2^q. \end{aligned}$$

Introducing (3.12) to the above and then dividing both sides by $1 - \delta_{s+a}$, with (3.6), we get

$$\|\bar{D}\Psi_{T_{01}}^* h\|_2^q \leq \iota^{q/2} \Delta a^{q/2-1} \|\Psi_{T^c}^* h\|_q^q + \|A\bar{D}\Psi^* h\|_q^q / (1 - \delta_{s+a}),$$

which is equivalent to

$$\|\bar{D}\Psi_{T_{01}}^* h\|_2^2 \leq \iota \Delta^{2/q} a^{1-2/q} \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2} \|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2} (1 + \delta_a)} \right)^{2/q}. \quad (3.17)$$

Introducing (3.16) to (3.17), one can get the desired result. \square

Now, we give an upper bound on $\|\Psi_{T_{01}}^* h\|_2^q$. By developing a relationship between $\|\Psi_{T_{01}}^* h\|_2^2$ and $\sum_{k=1}^\iota \|D_k D_{kT_{01}^k}^* h_k\|_2^2$, and then applying the above lemma, one can prove the following result.

Lemma 3.5. *Under the assumptions of Lemma 3.4, assume that $U < 1$. Then, we have*

$$\|\Psi_{T_{01}}^* h\|_2^q \leq \left(\frac{U + \iota \Delta^{2/q} + \sqrt{(U - \iota \Delta^{2/q})^2 + 4\iota \Delta^{2/q}}}{2(1 - U)} \right)^{q/2} \frac{1}{a^{1-q/2}} \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2} \|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2} (1 + \delta_a)} \right). \quad (3.18)$$

Proof. Note that by applying Cauchy-Schwarz inequality twice,

$$\begin{aligned} \|\Psi_{T_{01}}^* h\|_2^4 &= \left(\sum_{k=1}^\iota \|D_{kT_{01}^k}^* h_k\|_2^2 \right)^2 = \left(\sum_{k=1}^\iota \langle D_k D_{kT_{01}^k}^* h_k, h_k \rangle \right)^2 \\ &\leq \left(\sum_{k=1}^\iota \|h_k\|_2 \|D_k D_{kT_{01}^k}^* h_k\|_2 \right)^2 \leq \left(\sum_{k=1}^\iota \|h_k\|_2^2 \right) \left(\sum_{k=1}^\iota \|D_k D_{kT_{01}^k}^* h_k\|_2^2 \right) \\ &\leq \|h\|_2^2 \left(\sum_{k=1}^\iota \|D_k D_{kT_{01}^k}^* h_k\|_2^2 \right). \end{aligned}$$

Applying (3.13), we get

$$\|\Psi_{T_{01}}^* h\|_2^4 \leq (\|\Psi_{T_{01}}^* h\|_2^2 + a^{1-2/q} \|\Psi_{T^c}^* h\|_q^2) \left(\sum_{k=1}^\iota \|D_k D_{kT_{01}^k}^* h_k\|_2^2 \right).$$

Introducing (3.15),

$$\|\Psi_{T_{01}}^* h\|_2^4 \leq (\|\Psi_{T_{01}}^* h\|_2^2 + a^{1-2/q} \|\Psi_{T^c}^* h\|_q^2) (\iota \Delta^{2/q} a^{1-2/q} \mathcal{X} + U \|\Psi_{T_{01}}^* h\|_2^2),$$

where for notational simplicity, we set

$$\mathcal{X} = \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2} \|AD\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right)^{2/q}. \quad (3.19)$$

Rearranging terms, this can be rewritten as

$$(1-U)\|\Psi_{T_{01}}^* h\|_2^4 - (U\|\Psi_{T^c}^* h\|_q^2 + \iota\Delta^{2/q}\mathcal{X})a^{1-2/q}\|\Psi_{T_{01}}^* h\|_2^2 - \iota a^{2-4/q}\|\Psi_{T^c}^* h\|_q^2 \Delta^{2/q}\mathcal{X} \leq 0.$$

Noting that $U < 1$ and by solving a quadratic inequalities of type of $c_1 x^2 - c_2 x - c_3 \leq 0$ with variable $x \in [0, \infty)$ and positive constants c_1, c_2, c_3 , we get

$$\|\Psi_{T_{01}}^* h\|_2^2 \leq \frac{(U\|\Psi_{T^c}^* h\|_q^2 + \iota\Delta^{2/q}\mathcal{X}) + \sqrt{(U\|\Psi_{T^c}^* h\|_q^2 + \iota\Delta^{2/q}\mathcal{X})^2 + 4\iota(1-U)\|\Psi_{T^c}^* h\|_q^2 \Delta^{2/q}\mathcal{X}}}{2(1-U)a^{2/q-1}}.$$

Upper bounding the term $\|\Psi_{T^c}^* h\|_q^2$ by \mathcal{X} ,

$$\begin{aligned} \|\Psi_{T_{01}}^* h\|_2^2 &\leq \frac{(U + \iota\Delta^{2/q})\mathcal{X} + \sqrt{(U + \iota\Delta^{2/q})^2 \mathcal{X}^2 + 4\iota(1-U)\Delta^{2/q}\mathcal{X}^2}}{2(1-U)a^{2/q-1}} \\ &= \frac{U + \iota\Delta^{2/q} + \sqrt{(U - \iota\Delta^{2/q})^2 + 4\iota\Delta^{2/q}}}{2(1-U)a^{2/q-1}} \mathcal{X}, \end{aligned}$$

which leads to our desired result by introducing (3.19) and taking the $(q/2)$ -th power on both sides. \square

In what follows, we shall bound $\|\Psi_{T^c}^* h\|_q^q$ and $\|A\bar{D}\Psi^* h\|_q^q$. We first need the following result, which shows that the (\bar{D}, q) -RIP implies that the matrix $A\bar{D}\Psi^*$ satisfies robust Ψ -NSP $_q$.

Lemma 3.6 (Robust Ψ -NSP $_q$). *Under the assumptions of Lemma 3.4, we have*

$$\|\Psi_T^* h\|_q^q \leq \tilde{\theta} \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2} \|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right), \quad (3.20)$$

where

$$\tilde{\theta} = \left(\frac{U + \iota\Delta^{2/q} + \sqrt{(U - \iota\Delta^{2/q})^2 + 4\iota\Delta^{2/q}}}{2(1-U)} \right)^{q/2} \rho^{1-q/2}. \quad (3.21)$$

In particular, if

$$\iota\Delta^{2/q}(\rho^{2/q-1} + 1)\rho^{2/q-1} + U(1 + \rho^{2/q-1}) < 1, \quad (3.22)$$

then $\tilde{\theta} < 1$.

Proof. Note that by applying (2.1), one has

$$\|\Psi_{T_{01}}^* h\|_2^q \geq \|\Psi_T^* h\|_2^q \geq \|\Psi_T^* h\|_q^q / s^{1-q/2}.$$

Combining with (3.18), we get

$$\frac{\|\Psi_T^* h\|_q^q}{s^{1-q/2}} \leq \left(\frac{U + \iota\Delta^{2/q} + \sqrt{(U - \iota\Delta^{2/q})^2 + 4\iota\Delta^{2/q}}}{2(1 - U)} \right)^{q/2} \frac{1}{a^{1-q/2}} \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2} \|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1 + \delta_a)} \right),$$

which is equivalent to (3.20).

It remains to prove $\tilde{\theta} < 1$. This can be verified by showing that $\tilde{\theta}^{2/q} < 1$, which is guaranteed (since $U < 1$ by our assumptions) provided that

$$\sqrt{(U - \iota\Delta^{2/q})^2 + 4\iota\Delta^{2/q}} < 2\rho^{1-2/q}(1 - U) - (U + \iota\Delta^{2/q}).$$

Note that under assumption (3.22), the right hand side is always positive. Taking the second power on both sides, rearranging terms, a sufficient condition for the above is

$$(U - \iota\Delta^{2/q})^2 + 4\iota\Delta^{2/q} - (U + \iota\Delta^{2/q})^2 < 4\rho^{2-4/q}(1 - U)^2 - 4\rho^{1-2/q}(1 - U)(U + \iota\Delta^{2/q}),$$

which can be rewritten as

$$4\iota\Delta^{2/q}(1 - U) < 4\rho^{2-4/q}(1 - U)[1 - U(1 + \rho^{2/q-1})] - 4\iota\rho^{1-2/q}(1 - U)\Delta^{2/q}.$$

Dividing both sides by $4(1 - U)\rho^{2-4/q}$ and rearranging terms, a sufficient condition for the above is (3.22). From the above analysis, we have $\tilde{\theta} < 1$. The proof is finished. \square

Now, we shall choose $h = \hat{f} - f$, where \hat{f} is a solution of (3.3) and f is the “original signal” given by (3.3). Let Ω be the index set of the s largest entries of $|\Psi^* f|$. The following

results can be verified by using the fact that \hat{f} is a solution of (2.3). By a similar argument as that for Lemma 2.11, one gets that

$$\|\Psi_{T^c}^* h\|_q^q \leq \|\Psi_T^* h\|_q^q + 2\|\Psi_{\Omega^c}^* f\|_q^q \quad (3.23)$$

and

$$\|A\bar{D}\Psi^* h\|_q^q \leq m^{1-q/r}(2\varepsilon)^q. \quad (3.24)$$

We may now conclude the proof of Theorem 3.2. Note that assumptions (3.4) and (3.5) imply (3.22) since U is given by (3.14). Therefore, by Lemma 3.6, we have $\tilde{\theta} < 1$. Combining (3.23) with (3.20),

$$\|\Psi_T^* h\|_q^q \leq \frac{\tilde{\theta}}{1-\tilde{\theta}} \left(2\|\Psi_{\Omega^c}^* f\|_q^q + \frac{a^{1-q/2}\|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right), \quad (3.25)$$

and

$$\|\Psi_{T^c}^* h\|_q^q \leq \frac{1}{1-\tilde{\theta}} \left(2\|\Psi_{\Omega^c}^* f\|_q^q + \frac{\tilde{\theta}a^{1-q/2}\|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right). \quad (3.26)$$

By (3.13), (3.18) and noting that $\tilde{\theta}$ is given by (3.21), and ρ is given by (3.6),

$$\begin{aligned} \|h\|_2^2 &\leq \|\Psi_{T_{01}}^* h\|_2^2 + a^{1-2/q}\|\Psi_{T^c}^* h\|_q^2 \\ &\leq \theta^{2/q}s^{1-2/q} \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2}\|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right)^{2/q} + \rho^{2/q-1}s^{1-2/q}\|\Psi_{T^c}^* h\|_q^2 \\ &\leq \frac{\theta^{2/q} + \rho^{2/q-1}}{s^{2/q-1}} \left(\|\Psi_{T^c}^* h\|_q^q + \frac{a^{1-q/2}\|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right)^{2/q}. \end{aligned}$$

Introducing (3.26), and then applying (3.24) and a basic inequality $(b+c)^{1/q} \leq 2^{1/q-1}(b^{1/q} + c^{1/q})$, $\forall b, c \geq 0$, with ρ given by (3.6),

$$\begin{aligned} \|h\|_2^2 &\leq \frac{\tilde{\theta}^{2/q} + \rho^{2/q-1}}{(1-\tilde{\theta})^{2/q}s^{2/q-1}} \left(2\|\Psi_{\Omega^c}^* f\|_q^q + \frac{(1+\tilde{\theta})a^{1-q/2}\|A\bar{D}\Psi^* h\|_q^q}{\iota^{q/2}(1+\delta_a)} \right)^{2/q} \\ &\leq \frac{\tilde{\theta}^{2/q} + \rho^{2/q-1}}{(1-\tilde{\theta})^{2/q}s^{2/q-1}} \left(2\|\Psi_{\Omega^c}^* f\|_q^q + \frac{(1+\tilde{\theta})a^{1-q/2}m^{1-q/r}(2\varepsilon)^q}{\iota^{q/2}(1+\delta_a)} \right)^{2/q} \\ &\leq \frac{(\tilde{\theta}^{2/q} + \rho^{2/q-1})}{(1-\tilde{\theta})^{2/q}} \left(\frac{2^{2/q-1}\|\Psi_{\Omega^c}^* f\|_q}{s^{1/q-1/2}} + \frac{2^{1/q}(1+\tilde{\theta})^{1/q}\rho^{1/2-1/q}m^{1/q-1/r}\varepsilon}{\sqrt{\iota}(1+\delta_a)^{1/q}} \right)^2. \end{aligned}$$

Taking the square power on both sides, we get

$$\|h\|_2 \leq \frac{2^{2/q-1}(\tilde{\theta}^{1/q} + \rho^{1/q-1/2})}{(1 - \tilde{\theta})^{1/q}} \frac{\|\Psi_{\Omega^c}^* f\|_q}{s^{1/q-1/2}} + \frac{2^{1/q} \rho^{1/2-1/q} (1 + \tilde{\theta})^{1/q} (\tilde{\theta}^{1/q} + \rho^{1/q-1/2})}{\sqrt{\iota} (1 - \tilde{\theta})^{1/q} (1 + \delta_a)^{1/q}} m^{1/q-1/r} \varepsilon,$$

which leads to the desired result (3.7). To prove (3.8), we first apply (3.25), (3.26) and then use (3.24) to get

$$\begin{aligned} \|\Psi^* h\|_q^q &= \|\Psi_T^* h\|_q^q + \|\Psi_{T^c}^* h\|_q^q \\ &\leq \frac{2(1 + \tilde{\theta})}{1 - \tilde{\theta}} \|\Psi_{\Omega^c}^* f\|_q^q + \frac{2\tilde{\theta}}{\iota^{q/2} (1 - \tilde{\theta}) (1 + \delta_a)} a^{1-q/2} \|A\bar{D}\Psi^* h\|_q^q \\ &\leq \frac{2(1 + \tilde{\theta})}{1 - \tilde{\theta}} \|\Psi_{\Omega^c}^* f\|_q^q + \frac{2^{q+1}\tilde{\theta}}{\iota^{q/2} (1 - \tilde{\theta}) (1 + \delta_a)} a^{1-q/2} m^{1-q/r} \varepsilon^q, \end{aligned}$$

which leads to the desired result (3.8). The proof is finished.

Remark 3.7. 1) In the proof, we have proved that suitable conditions on the modified q -RIP and the mutual coherence between the dictionaries D_1, D_2, \dots, D_ι imply that $A\bar{D}\Psi$ satisfies Ψ -NSP $_q$, i.e.,

$$\|\Psi_T^* h\|_q^q \leq \tilde{\theta} \|\Psi_{T^c}^* h\|_q^q,$$

for all $h \in \ker(A\bar{D}\Psi)$ and all $T \subset [d]$ with $|T| \leq s$, where $\tilde{\theta} < 1$. Now we can define the null space property for compressed data separation as follow: For $q \in (0, 1]$, A is said to satisfy the l_q -split null space property with respect to dictionaries D_1, D_2, \dots, D_ι of order s if there exists some $\theta \in [0, 1)$ such that

$$\sum_{k=1}^{\iota} \|D_{kT_k}^* h_k\|_q^q \leq \theta \sum_{k=1}^{\iota} \|D_{kT_k^c}^* h_k\|_q^q,$$

for all h_1, h_2, \dots, h_ι such that $(\sum_{k=1}^{\iota} h_k) \in \ker(A)$ and all $T_1 \subset [d_1], T_2 \subset [d_2], \dots, T_\iota \subset [d_\iota]$ with $\sum_{k=1}^{\iota} |T_k| \leq s$. Note that the null space property for standard compressed sensing is one of the well known conditions on measurement matrices (e.g. [24, 13, 6]). Here, we provide a definition of null space property for compressed data separation, which may be of independent interest.

2) From the proof, we see that the following inequality holds

$$\|\Psi^* \hat{f} - \Psi^* f\|_2 \leq C_1 \frac{\|\Psi^* f - (\Psi^* f)_{[s]}\|_q}{s^{1/q-1/2}} + C_2 m^{1/q-1/r} \varepsilon.$$

4 Numerical realization and discussion

In this final section, we discuss numerical realization of the constrained l_q analysis (P_q), and provide further discussions on our theoretical analysis.

The constrained l_q analysis problem (P_q) proposed to recover f is nonconvex. Due to its nonconvexity, finding a global minimizer of problem (P_q) is generally NP-hard. We thus solve such a nonconvex problem by solving a sequence of convex problems, as often done in standard compressed sensing for standard l_q minimization, e.g., [22, 15]. The first possible method is the iteratively reweighted l_1 analysis which iteratively solves the following weighted l_1 analysis (IRL1)

$$f^{j+1} = \arg \min_{\tilde{f} \in \mathbb{R}^n} \sum_{i=1}^d \omega_i^j |\langle d_i, \tilde{f} \rangle| \quad \text{subject to} \quad A\tilde{f} = y,$$

where d_i is the i -th column of D , $\omega_i^j = (|\langle d_i, f^j \rangle| + \varsigma_j)^{q-1}$ and (ς_j) a nonincreasing sequence of positive numbers. Another potential method is iteratively reweighted least squares (IRLS), which iteratively solves the following weighted least square problem

$$f^{j+1} = \arg \min_{\tilde{f} \in \mathbb{R}^n} \sum_{i=1}^d \omega_i^j |\langle d_i, \tilde{f} \rangle|^2 \quad \text{subject to} \quad A\tilde{f} = y,$$

where $\omega_i^j = (|\langle d_i, f^j \rangle|^2 + \varsigma_j)^{q/2-1}$. The key of the above methods lie in that the object functions in both methods are approximations to $\|D^* \tilde{f}\|_q^q$ when $\varsigma \rightarrow 0$. In fact, one may derive convergence results for both these methods using the techniques developed in this paper and in [22, 15]. We postpone the details for a future work. We here provide a simple simulation, demonstrating that the analysis IRLS can solve the constrained l_q analysis (P_q)

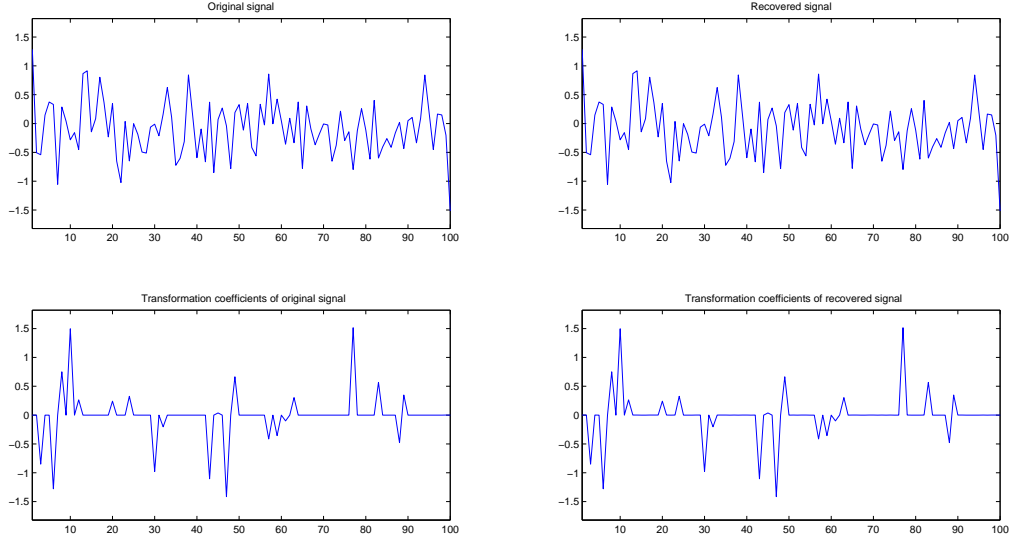


Figure 1: Reconstruction via analysis IRLS

efficiently. In this simulation, we let $n = 100$, $d = 110$, $m = 50$, $q = 0.7$ and the sparsity of D^*f as $s = 25$. The entries in the $m \times n$ measurement matrix A were randomly generated according to a normal distribution. The $n \times d$ matrix d is a random tight frame, generated by the approach in [33]. Figure 1 shows that the analysis IRLS reconstructs the signal f exactly.

In this paper, we discussed sparse recovery with general frames from random measurements via the l_q -analysis optimization (P_q) with $0 < q \leq 1$. We introduced a notion of (D, q) -RIP. It is a natural extension of the standard q -RIP defined by Chartrand and Staneva [12] for standard compressed sensing, and is different from the D -RIP defined in (1.2). We established an (D^\dagger, q) -RIP guarantee result for the l_q -analysis optimization (P_q) . We proved the result by investigating the relationship between the (D^\dagger, q) -RIP constant and the D -NSP $_q$ constant, which at the same time may shed some lights on how to establish a tighter relationship between the D^\dagger -RIP constant and the D -NSP $_q$ constant. Subsequently, we showed

how many random Gaussian measurements are needed for the (D^\dagger, q) -RIP condition to hold with high probability. Finally, we discussed compressed data separation by using the introduced (D, q) -RIP. We showed that under an usual assumption that the two dictionaries satisfy a mutual coherence condition, the l_q split analysis with $0 < q \leq 1$ can approximately reconstruct the distinct components from fewer random Gaussian measurements with small q than when $q = 1$. Our results provide theoretical basis for further designing algorithm to solve (P_q) and the l_q split analysis optimization. Our proof techniques may shed some lights on improving the previous D -RIP guarantee results. Further issues are to design numerical methods (e.g., iteratively reweighted method) to find approximate solutions of the l_q -analysis optimization applied in practical applications, and to consider other random measurements instead of Gaussian measurements.

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